Indian Statistical Institute, Bangalore Centre
M.Math. (I Year) : 2015-2016
Semester I : Backpaper Examination
Measure Theoretic Probability

Dec. 2015 Time: 3 hours. Maximum Marks : 100

Note: Notation and terminology are understood to be as used in class. State clearly the results you are using in your answers.

1. (15 marks) Let $\Omega = \mathbb{R}$, and $\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra of $\mathbb{R}$. Let $\mathcal{K} = \{[a,b] : a \leq b, a, b \in \mathbb{R}\}$. Show that $\sigma(\mathcal{K}) = \mathcal{B}(\mathbb{R})$. (Here $\sigma(\mathcal{K})$ denotes the smallest σ-algebra containing $\mathcal{K}$.)

2. (10 +10 = 20 marks) (i) Let $F(\cdot)$ be a distribution function on $\mathbb{R}$. Show that there exists a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, and a real valued random variable X such that $F(\cdot)$ is the distribution function of X.

   (ii) Let $f(\cdot)$ be a probability density function on $\mathbb{R}$. Show that there exists a real valued random variable X such that $f(\cdot)$ is the probability density function of X.

3. (15 + 15 + 10 = 40 marks) (i) Let X be a real valued random variable with characteristic function φ. Suppose X has finite expectation. Show that φ is differentiable, with a uniformly continuous derivative.

   (ii) Let μ_a denote the distribution of a random variable having exponential distribution with parameter $a > 0$. Find the characteristic function of μ_a. Using (i), find the expectation of the exponential distribution with parameter a.

   (iii) Let $a_n \to a$, where $a_n, a > 0$. Using (ii), show that $\mu_{a_n} \Rightarrow \mu_a$, as $n \to \infty$.

4. (25 marks) Let $\{X_n : n = 1, 2, \cdots\}$ be a sequence of independent identically distributed real valued random variables with strictly positive density function. Let $F(\cdot)$ denote the common distribution function. Show that there exist constants $m \in \mathbb{R}$, $\sigma^2 > 0$ such that for any $y \in \mathbb{R}$,

$$\lim_{n\to\infty} P\Big(\frac{1}{\sigma\,\sqrt{n}}[F(X_1) + \cdots + F(X_n) - nm] \;\leq\; y\Big) = \Phi(y),$$

where $\Phi(\cdot)$ denotes the distribution function of the standard normal distribution. Also find the constants m, σ^2.